CALCULUS 2 NOTEBOOK

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متسلسلة المتناوبة

Alternating series and Absolute Convergance

الدراسة في الرياضيات تكون عملية منظمة لا فوضوية (د. حسن العزة)

A series of the form

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n \quad , b_n > 0 \quad \text{or} \quad \sum_{n=1}^{\infty} (-1)^{n} b_n \quad , b_n > 0$$

is called an oltarnating series.

Examples

[1] $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$

This is an oltarnating series with $b_n = \frac{1}{n} > 0$

[2] $-1 + \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} - \frac{1}{2^4} + \cdots$

This is an oltarnating series with $b_n = \frac{1}{2^{n-1}} > 0$

[3] $1 + \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2} - \frac{1}{8^2} - \cdots$

This is not ~~an~~ oltrnating series.

The altrnating series test

Given an alternating series with $b_n > 0$
Then the series is convergent if:-

[1] $\{b_n\}_{n=1}^{\infty}$ is decreasing.

[2] $\lim_{n\to\infty} b_n = 0$

Example

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} - -$$

This is an alternating series $b_n = \frac{1}{n} > 0$

(1) $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is decreasing (check!!) :)

[2] $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$

By altrnating series test, the series is convergent.

Absolute Convergence :-

Let $\sum_{n=1}^{\infty} a_n$ be a series

Consider $\sum_{n=1}^{\infty} |a_n|$

then - if $\sum |a_n|$ is convergent then we say that the series $\sum a_n$ is absolutly convergent.

- if $\sum |a_n|$ is divergent then we say that the series $\sum a_n$ divergent absolutly.

Notes

[1] If $\sum_{n=1}^{\infty} a_n$ is absolutly convergent then this series is convergent

[2] If $\sum_{n=1}^{\infty} a_n$ is absolutly divergent then this series might be convergent or divergent.

[3] If $\sum_{n=1}^{\infty} a_n$ is absolutly divergent then it is convergent then we say this series conditionaly convergent.

* Test for Convergent the series.

1) $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}-\frac{1}{7^2}-\frac{1}{8^2}+\dots$

$$\sum_{n=1}^{\infty}|a_n| = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\dots$$

This is a p-series with $p=2>1$, the series is Convergent

So, the series $\sum a_n$ is absolutly Convergent so, it is Convergent

2) $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\dots$

$$\sum_{n=1}^{\infty}|a_n| = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\dots$$

This series is a p-series with $p=1$, then it is divergent, hence the series $\sum a_n$ is absolutly divergent, but this is an alternating series with $b_n=\frac{1}{n}>0$

$$\{b_n\}_{n=1}^{\infty} = \left\{\frac{1}{n}\right\}_{n=1}^{\infty} \text{ divergent } \{\text{check}\} \Rightarrow \lim b_n = 0$$

by alternating series test, the series $\sum a_n$ is Convergent, then the series $\sum a_n$ is conditionaly Convergent.

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{n+1}$, Test for absolutly Convergent Conditionaly Convergent or divergent

Consider $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{n}{n+1}$

$\lim_{n \to \infty} \frac{n}{n+1} = 1 \neq 0$ by divergence test the series $\sum |a_n|$ is divergent, So $\sum_{n=1}^{\infty} a_n$ is absolutly divergent

although the series $\sum \frac{(-1)^{n-1} n}{n+1}$ alternating with $b_n = \frac{n}{n+1} \Rightarrow \lim b_n = 1 \neq 0$ we can apply the alternating series test

Consider $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{n+1} \rightarrow a_n = \frac{(-1)^{n-1} n}{n+1}$

$\lim a_n$ does not exist $\rightarrow$ by divergence test this series is divergent.

* Test for Convergent $\sum_{n=1}^{\infty} \frac{\sin^3(n)}{n^3}$

Consider $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left| \frac{\sin^3(n)}{n^3} \right|$

$$0 \leq \left| \frac{\sin^3(n)}{n^3} \right| \leq \frac{1}{n^3}$$

$\sum \frac{1}{n^3}$ is a p-series with $p = 3 > 1$, so it's Conv.
- by Comparison test the series $\sum_{n=1}^{\infty} |a_n|$ is Convergent then $\sum_{n=1}^{\infty} \frac{\sin^3(n)}{n^3}$ is absolutly Convergent and hence it's Convergent test.

* $\sum_{n=1}^{\infty} (-1)^n \left(1+\frac{1}{n}\right)^{n^2}$ يوجد حد سالب (نأخذ الـabsolute)

Consider $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2}$

$\lim_{n \to \infty} a_n^{\frac{1}{n}} = e$ (check), so, the series is divergent, So, the series $\sum \left(1+\frac{1}{n}\right)^{n^2}$ is divergent, so $\sum (-1)^n \left(1+\frac{1}{n}\right)^{n^2}$ is absolutly divergent

Power series

A power series in x is a formal sum of the form $a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

where $a_0, a_1, a_2, a_3, \ldots$ are real number called the coefficient to the power series.

If we substitae instead x a real number $x = r$, we get the ordinary series

$$a_0 + a_1 r + a_2 r^2 + a_3 r^3 + \cdots$$

Examples :-

[1] $1 + x + x^2 + x^3 + \cdots$

This is a power series in x with $C_0 = 1$, $C_1 = 1$, $C_2 = 1$, $C_3 = 1, \ldots$

[2] $$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

This is a power series in x with $C_0 = 1$

$C_1 = \frac{1}{1!}$, $C_2 = \frac{1}{2!}$, $C_3 = \frac{1}{3!}$, $\ldots$, $C_n = \frac{1}{n!}$

$\boxed{3}$ $\sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)!} = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!}, \dots$

This is a power series in x with

$C_0 = 0$, $C_1 = \frac{1}{1!}$, $C_2 = 0$, $C_3 = \frac{1}{3!}$, $C_4 = 0$ ...

x^0 معامل

x^2 معامل

$$C_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{1}{n!} & \text{if } n \text{ is odd} \end{cases}$$

☆ more generally, a Power series in $x-a$ is a power series a formal sum of the form

$$C_0 + C_1(x-a) + C_2(x-a)^2 + C_3(x-a)^3 + \dots$$

For $a=0$, we get a power series in x

Ex $\sum_{n=1}^{\infty} \frac{(x-1)^n}{3^n(n+1)}$

$= \frac{(x-1)}{3(2)} + \frac{(x-1)^2}{3^2(2+1)} + \frac{(x-1)^3}{3^3(3+1)} + \cdots$

This is apower series in $x-a$, is

$\boxed{a=1}$

Ex $\sum_{n=0}^{\infty} (x+1)^n$

$= 1 + (x+1) + (x+1)^2 + (x+1)^3 + \cdots$

This is apower series in $x+a$, with $a=-1$

Interval of Convergence of a power Series.

* The interval of convergence of a power series is the set of all values of X in the power series for wich the obtained series after subsituting this value of convergent.

* How to find the interval of Convergence of a power series?

دائماً بيجي سؤال عليها بالفاينل !!

Given a power series in $X-a$, then the interval of convergence of this power series is exactly one of the following three cases:-

- The interval of convergence is a single point, namely $\{a\}$.
- The interval of convergent is the set $-\infty < X < \infty$
- There is a positive real number R, such that the power series is convergent on $(a-R, a+R)$, and it is divergent for all $X > R+a$ and $X < a-R$.

At the end point ~~might~~ may be convergent or divergent.

* Find the interval of convergence of the following series

1. $\sum_{n=1}^{\infty} n!\,(x-1)^n$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim \left|\frac{(n+1)!\,(x-1)^{n+1}}{n!\,(x-1)^n}\right|$$

$$= |x-1| \lim_{n\to\infty} |n+1|$$

$$= \infty \qquad \text{unless } x-1=0$$

$$x=1$$

So, the interval of convergence is $x=1$ only.

2. $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim \left|\frac{(x-3)^{n+1}}{n+1}\cdot\frac{n}{(x-3)^n}\right|$$

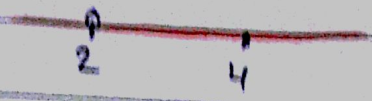
$$= \lim_{n\to\infty}\left|\frac{(x-3)\,n}{n+1}\right|$$

$$= |x-3| \lim_{n\to\infty}\left|\frac{n}{n+1}\right|$$

$$= |x-3| < 1$$

$$-1 < x-3 < 1$$

$$2 < x < 4$$



داخل الفترة Conv. خارجها لا تك.
عدا الحدود

At $x=2$ we get

$$\sum_{n=1}^{\infty} \frac{(2-3)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

this is a convergent series by the alternating series test.

At $x=4$ we get

$$\sum_{n=1}^{\infty} \frac{(4-3)^n}{n} = \sum_{n=1}^{\infty} \frac{(1)^n}{n}$$

this is divergent series becaus it is p-series with $p=1$

Interval of convergent $= [2,4)$

$R = \frac{4-2}{2} = 1 \rightsquigarrow$ Radius of convergence

تعريف قطر التقارب:

$\boxed{3}$ $\sum_{n=1}^{\infty} \frac{x^n}{n!}$

$$\lim \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\cancel{x^{n+1}}^{\,x}}{\cancel{(n+1)!}_{\,n+1}} \cdot \frac{\cancel{n!}}{\cancel{x^n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x}{n+1} \right|$$

$$= |x| \cdot \lim_{n\to\infty} \left| \frac{1}{n+1} \right|$$

$$= 0 < 1$$

دائماً تكون Conv.

Interval of Convergence $= (-\infty, \infty)$

Representation of afunction by apower series.

Consider the power series

$1+x+x^2+x^3+x^4+\ldots$

find the interval of Convergence

$\lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \left|\frac{x^{n+1}}{x^n}\right|$

$|x| \leq 1$

$-1 < x < 1$

At $x=-1$ we get

$1-1+1-1+1-1$ divergent

Interval of Convergent $-1<x<1$

$(-1,1)$

In fact this is a geometric series with sum =

$\frac{1}{1-x} = 1+x+x^2+x^3+\ldots$

$-1 < x < 1$

Represent the following function as a power series in x;

1) $f(x) = \frac{1}{1+x} = \frac{1}{1-(-x)}$

$= 1 + -x + (-x)^2 + (-x)^3 + (-x)^4 + \cdots$

$-1 < -x < 1$

$\cancel{-1 < x < 1}$

$1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$

$-1 < x < 1$

2) $f(x) = \frac{1}{2-x}$

$= \frac{1}{2(1-\frac{x}{2})} = \frac{1}{2}\left(1 + \frac{x}{2} + \left(\frac{x}{2}\right)^2 + \left(\frac{x}{2}\right)^3 + \left(\frac{x}{2}\right)^4 + \cdots\right)$

$-1 < \frac{x}{2} < 1$

$= \frac{1}{2} + \frac{x}{2^2} + \frac{x^2}{2^3} + \frac{x^3}{2^4} + \cdots$

$-2 < x < 2$

$\sum_{n=1}^{\infty} \frac{x^{n-1}}{2^n}$

$\boxed{4}$ $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + (-x^2)^4 + \cdots$

$-1 < +x^2 < 1$

$= 1 - x^2 + x^4 - x^6 + \cdots$

$-1 < x < 1$

* $f(x) = \frac{1}{x}$

$= \frac{1}{1+x-1} = \frac{1}{1-(1-x)} = \frac{1}{1-(-(x-1))}$

$= 1 + (-(x-1)) + (-(x-1))^2 + (-(x-1))^3 + \cdots$

$-1 < x-1 < 1$

$\sum_{n=1}^{\infty} (-1)^{n-1} (x-1)^{n-1}$

$0 < x < 2$

Differentiation of power series.

Given a power series in $(X-a)$

$$f(x) = C_0 + C_1(x-a) + C_2(x-a)^2 + \cdots$$

$$X \in I$$

$$f'(x) = 0 + C_1 + 2C_2(x-a) + \cdots$$

$$x \in I$$

Ex $\frac{1}{1-X} = 1 + X + X^2 + X^3 + X^4 + \cdots$

$$-1 < X < 1$$

$$-(1-x)^{-2}(-1) = 0 + 1 + 2X + 3X^2 + \cdots$$

$$\frac{1}{(1-x)^2} = 1 + 2X + 3X^2 + 4X^3 + \cdots$$

$$-1 < X < 1$$

$$\sum_{n=1}^{\infty} n X^{n-1}$$

* Find the sum :-

مهم !!

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^n.$$

$$= \frac{1}{2} \left[\sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^{n-1} \right]$$

$$= \frac{1}{2} \cdot \frac{1}{\left(1-\frac{1}{2}\right)^2} \quad \text{since} \quad X = \frac{1}{2}$$

$$\frac{1}{(1-X)^2} = \sum_{n=1} n X^{n-1}$$

$$-1 < X < 1$$

$$= 2$$

* $$\sum_{n=2}^{\infty} \frac{n}{2^n}$$

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1}{2} + \sum_{n=2}^{\infty} \frac{n}{2^n}$$

$$2 = \frac{1}{2} + \sum_{n=2}^{\infty} \frac{n}{2^n}$$

the above result so $\sum_{n=2}^{\infty} \frac{n}{2^n} = \frac{3}{2}$

Integration of power series

Given $f(x) = C_0 + C_1(x-a) + C_2(x-a)^2 + \cdots$

$x \in I$

$$\int f(x)\,dx = C_0 x + \frac{C_1(x-a)^2}{2} + \frac{C_2(x-a)^3}{3} + \cdots$$

$x \in I$

Ex

Given that $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$

$-1 < x < 1$

represent the function $f(x) = \tan^{-1}(x)$ as a power series

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$

$$= 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \cdots$$

$-1 < -x^2 < 1$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$$

$-1 < x < 1$

$$\int \frac{1}{1+x^2} = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots \qquad -1 < x < 1$$

$$\tan^{-1}x + c = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad -1 < x < 1$$

لا تنسى الثابت

since $-1 < 0 < 1$

لايجاد قيمة C منه
نأخذ اي عدد داخل الفترة
ونعوضه

$\tan^{-1}(0) + C = 0 + 0 + 0 - -$

$\boxed{C = 0}$

$$\tan^{-1}(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1} \qquad -1 < x < 1$$

* Using this representation find the sum

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \left(\frac{1}{\sqrt{3}}\right)^{2n-1}}{2n-1} = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

Maclaurin Series   الامتحان النهائي لا يخلو من سؤال عليها

Let $y = f(x)$ be a function that is infinetly differentiable at $x = 0$, Then

[1] the maclaurin series of $y = f(x)$ is

$$f(0) + \frac{f'(0)}{1!} + \frac{f''(0)}{2!} + \frac{f'''(0)}{3!} + \frac{f^{(4)}(0)}{4!} + \cdots$$

[2] $$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

Ex Find the Maclaurin series of f

1. $f(x) = e^x$ $\rightsquigarrow$ معروف عند إيجاد المشتقة
موجودة ٨٨

$f(0) = 1$

$f'(x) = e^x$

$f'(0) = 1$

$f''(x) = e^x$

$f''(0) = 1$

$\vdots$

The maclaurin series is

$$1 + \frac{1}{1!}x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

to find the interval of convergence

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim\left|\frac{\cancel{x^{n+1}}^{\,x}}{\cancel{(n+1)!}_{\,n+1}} \cdot \frac{\cancel{n!}}{\cancel{x^n}}\right|$$

$$= |x| \lim_{n\to\infty}\left|\frac{1}{n+1}\right|$$

$$0 < 1$$

Interval of convergence $(-\infty, \infty)$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad -\infty < x < \infty$$

* find the sum $\sum_{n=0}^{\infty} \frac{2^n}{n!}$

$$= e^2$$

* $\sum_{n=1}^{\infty} \frac{2^n}{n!} = e^2 - 1$

* $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} = \frac{1}{e} - 1$

*Ex

$f(x) = \sin x$

Sol $\quad f(0) = 0$

$f'(x) = \cos x \rightsquigarrow f'(0) = 1$

$f''(x) = -\sin x \rightsquigarrow f''(0) = 0$

$f'''(x) = -\cos x \rightsquigarrow f'''(0) = -1$

$f^{(4)}(x) = \sin x \rightsquigarrow f^{(4)}(0) = 0$

← رجعت لنفس الدتقلات

(تتكرر كل اربع مرات)

The maclaurin series of $f(x) = \sin x$

is

$$0 + \frac{1}{1!}x + \cancel{\frac{0}{2!}x^2} + \frac{-1}{3!}x^3 + \cancel{\frac{0}{4!}x^4} + \frac{1}{5!}x^5 + \cdots$$

$$= \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{3!} - \frac{x^7}{7!} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)!}$$

Interval of Convergence $(-\infty, \infty)$ $\qquad$ * نفس interval of Convergence

Find the sum $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \pi^{2n-1}}{6^{2n}\,[(2n-1)!]}$

$$= \frac{1}{6} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \left(\frac{\pi}{6}\right)^{2n-1}}{(2n-1)!}$$

$$= \frac{1}{6} \sin\left(\frac{\pi}{6}\right) = \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}$$

$f(x) = \cos x$

* هناك طريقتين

① الحل مثل المثال السابق

② من السؤال السابق اوجدنا متسلسلة sin

وال sin هو مشتقة ال cos

فنشتق الطرفين

since

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

By Differntiation.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \qquad x \in \mathbb{R}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

* Find the maclaurin series of

$f(x) = \frac{1}{1-x}$, Then find the Maclaurin

series of $g(x) = \frac{x^2}{1+x^2}$

$\left\{ f(x) = (1-x)^{-1} \right.$

$f(0) = 1$

$f'(x) = -1(1-x)^{-2}(-1) = (1-x)^{-2}$

$f'(0) = 1$

$f''(x) = -2(1-x)^{-3}(-1) = 2(1-x)^{-3} = 2!\,(1-x)^{-3}$

$f''(0) = 2!$

$f'''(x) = -6(1-x)^{-4}(-1) = 6(1-x)^{-4} = 3!\,(1-x)^{-4}$

$f'''(0) = 3!$

$f^{(4)}(x) = 4!\,(1-x)^{-5}$

$f^{(4)}(0) = 4!$ ---

The Maclaurin series of $f(x) = \frac{1}{1-x}$

$= 1 + \frac{\cancel{1!}}{\cancel{1!}} x + \frac{\cancel{2!}}{\cancel{2!}} x^2 + \frac{\cancel{3!}}{\cancel{3!}} x^3 + \frac{\cancel{4!}}{\cancel{4!}} x^4 + \cdots$

$= 1 + x + x^2 + x^3 + x^4 + \cdots$

We find the interval of Convergence.

The interval of Convergence = $(-1,1)$

$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \qquad -1 < x < 1$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \cdots$$

$$= 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \cdots$$

$$-1 < -x^2 < 1$$

$$-1 < x < 1$$

so $\frac{x^2}{1+x^2} = x^2[1 - x^2 + x^4 - x^6 + x^8 + \cdots]$

$$-1 < x < 1$$

$$= x^2 - x^4 + x^6 - x^8 + x^{10} - \cdots$$

$$-1 < x < 1$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n+2} \qquad -1 < x < 1$$

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $-\infty < x < \infty$

$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$ $\qquad -\infty < x < \infty$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (x)^n}{n!} \qquad -\infty < x < \infty$$

$g(x) = \frac{x}{e^x} = xe^{-x} = \sum_{n=0}^{\infty} \frac{(-1) x^{n+1}}{n!}$, $-\infty < x < \infty$

Taylor series

Let $y=f(x)$ be a function that is infinitly differentiable at $x=a$,

Then the Taylor series of $y=f(x)$ at $x=a$ is

$$f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots$$

of the interval of convergence $=I$.

$$f(x)=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots$$

$x\in I$

* Ex

Find the Taylor series of

$f(x) = \ln(x)$ at $a=1$

sol

$f(1) = 0$

$f'(x) = \frac{1}{x} \rightsquigarrow f'(1) = 1$

$f''(x) = \frac{-1}{x^2} \rightsquigarrow f''(1) = -1$

$f'''(x) = +2x^{-3} \rightsquigarrow f'''(1) = 2 = 2!$

$f^{(4)}(x) = -6x^{-4} = -3!\,x^{-4} \rightsquigarrow f^{(4)}(1) = -(3!)$

$f^{(5)}(x) = 4!\,x^{-5} \rightsquigarrow f^{(5)}(1) = 4!$

$\vdots$

The Taylor series of $f(x) = \ln x$ at $a=1$ is

$$0 + \frac{1}{1!}(x-1) + \frac{-1}{2!}(x-1)^2 + \frac{2!}{3!}(x-1)^3 + \frac{-(3!)}{4!}(x-1)^4 + \cdots$$

$$= \frac{(x-1)}{1} - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots$$

Interval of Convergence $= [0,2)$ {check ☺}

So $\ln(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-1)^n}{n}$

$0 \leq x < 2$

* $g(x) = x\ln(x) \qquad = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\, x\, (x-1)^n}{n}$

The Binomial series.

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} b^{n-k} a^k$$

Maclaurin series of

$$f(x) = (1+x)^a \qquad , a \neq 0$$

is $f(x) = 1 + \frac{a}{1!}x + \frac{a(a-1)}{2!}x^2 + \frac{a(a-1)(a-2)}{3!}x^3 + \cdots$

$$-1 < x < 1$$

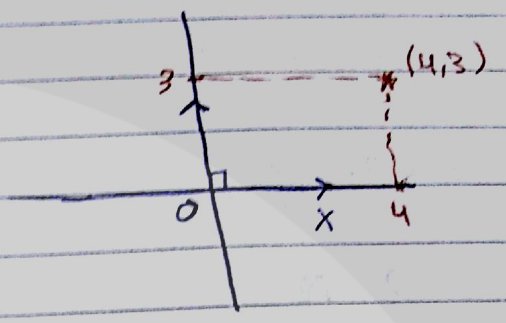
$$f(x) = (1+x)^{-1}$$

$$= 1 + \frac{-1}{1!}x + \frac{(-1)(-2)}{2!}x^2 + \frac{(-1)(-2)(-3)}{3!}x^3 + \cdots$$

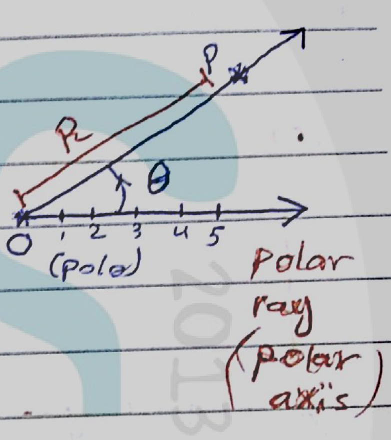
$$= 1 - x + x^2 - x^3 + \cdots \qquad -1 < x < 1$$

Polar Coordinates

* The xy Coordinates is Called sometimes the rectangular Coordinates.



* polar Coordinates.



θ is the angle by which the polar axis is rotated inorder to pass through P

θ is measured in radians

θ>0 :- θ is positive if Counter ~~clock~~ clockwise direction.

θ<0 :- when it is in the Clockwise direction.

R is the distance between the pole O and the point P.

So, P is determined by the ordered pair (r, θ)

حسبي الله ونعم الوكيل

Example

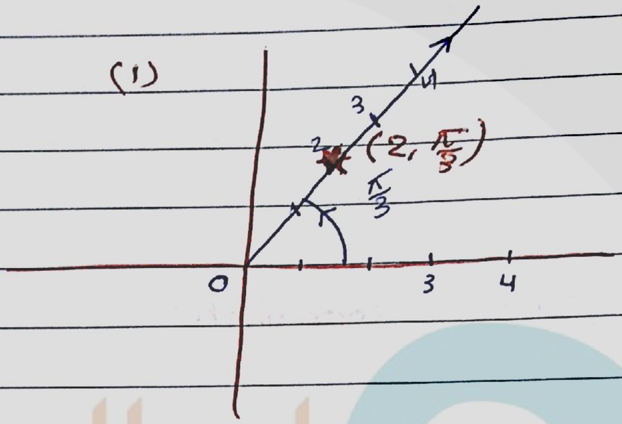
plot the points in polar Coordinates.

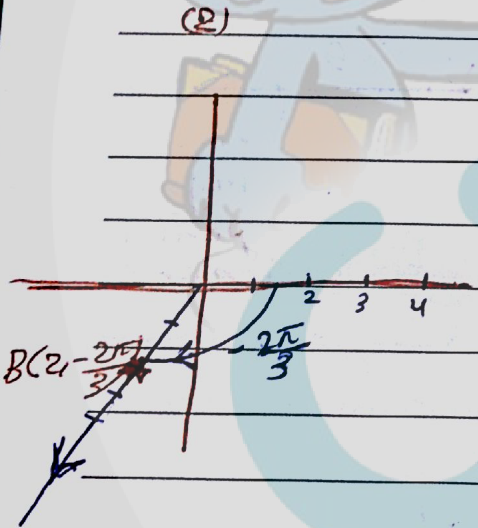
(1) $A\left(2, \frac{\pi}{3}\right)$

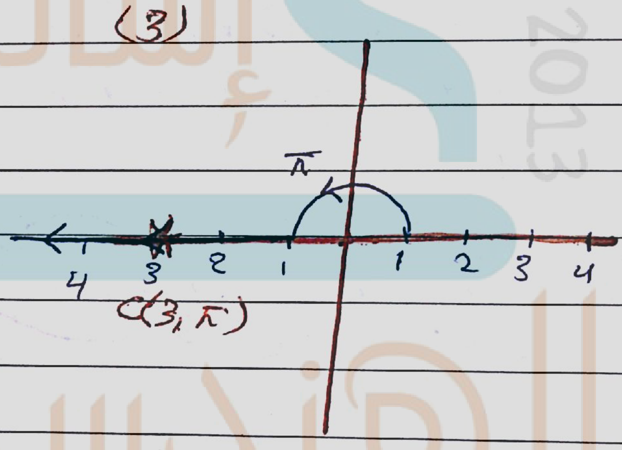
2) $B\left(2, -\frac{2\pi}{3}\right)$

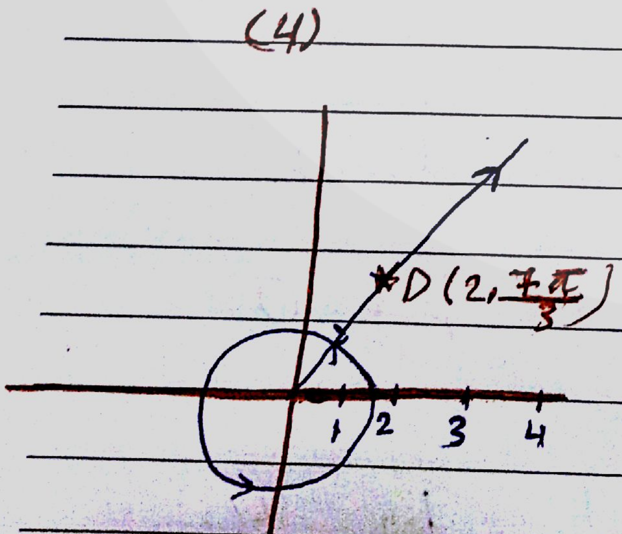
3) $C(3, \pi)$

4) $D\left(2, \frac{7\pi}{3}\right)$









$A\left(2, \frac{\pi}{3}\right)$ and $D\left(2, \frac{7\pi}{3}\right)$ represent the same point in polar Coordinates, such points are Called equivalent points.

* r in polar coordinates can be viewed to be negative to mean $|r|$ unit in the opposite direction of the polar ray.

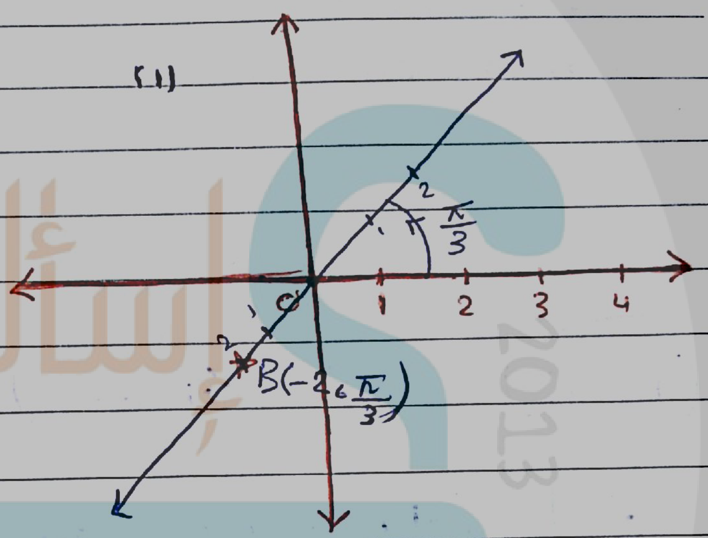
plot the point

1) $B\left(-2, \frac{\pi}{3}\right)$

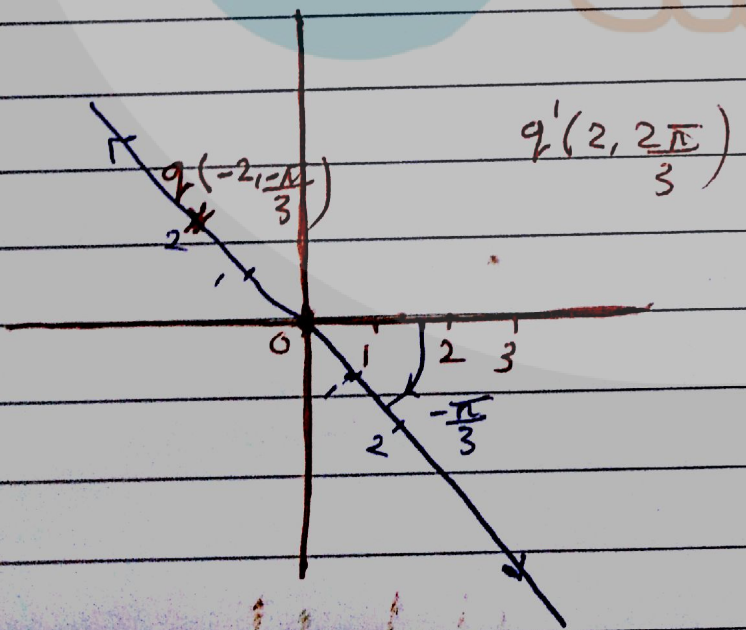
2) $q\left(-2, -\frac{\pi}{3}\right)$

* اولاً ندور الشعاع بالزاوية المعطاة
* نمد الشعاع على استقامته ابعد (السالب)

(1)



(2)



$q\left(-2, -\frac{\pi}{3}\right)$
$q'\left(2, \frac{2\pi}{3}\right)$
are equivalent.

* r in polar coordinates can be viewed to be negative to mean $|r|$ unit in the opposite direction of the polar ray.

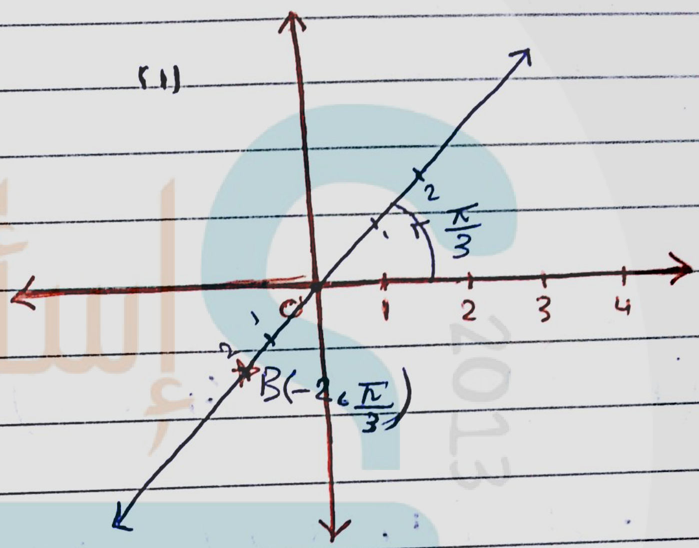
* plot the point

1) $B\left(-2, \frac{\pi}{3}\right)$

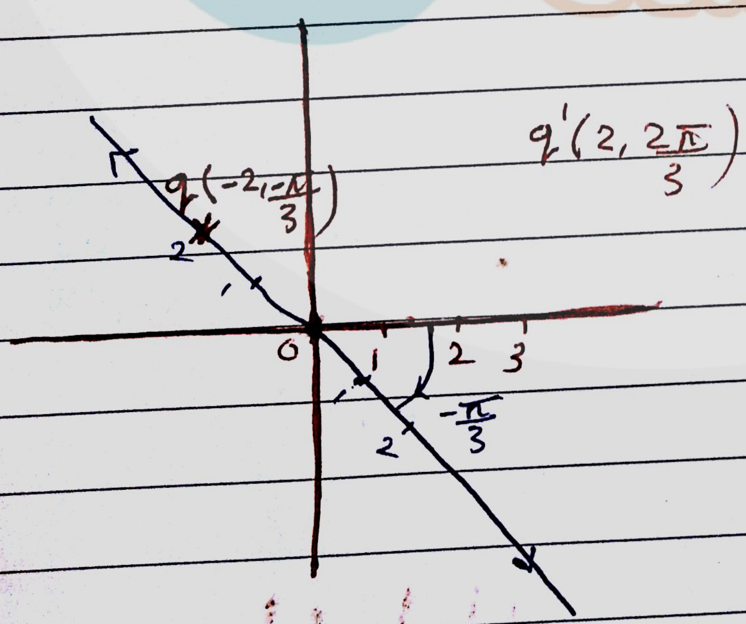
2) $q\left(-2, -\frac{\pi}{3}\right)$

* أولاً نعين الشعاع بالزاوية المعطاة
* نمد الشعاع على استقامته إلى الجانب السالب

(1)



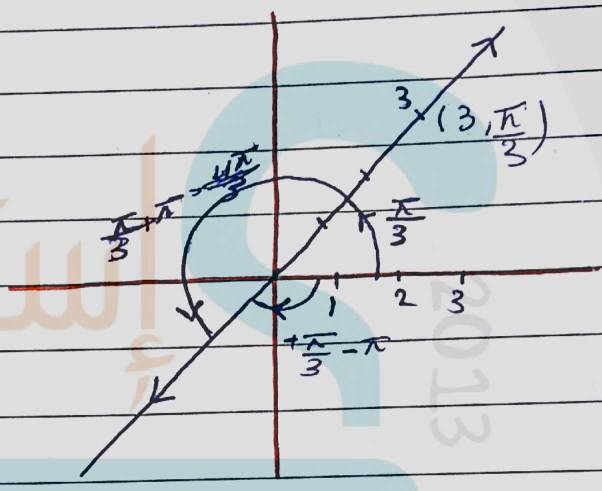
(2)



$q\left(-2, -\frac{\pi}{3}\right)$
$q'\left(2, \frac{2\pi}{3}\right)$
are equivalent.

* Example

Write down the polar Coordinates of the point $p(3, \frac{\pi}{3})$ in polar Coordinates So that ~~are~~ r is negative.



The Coordinates are

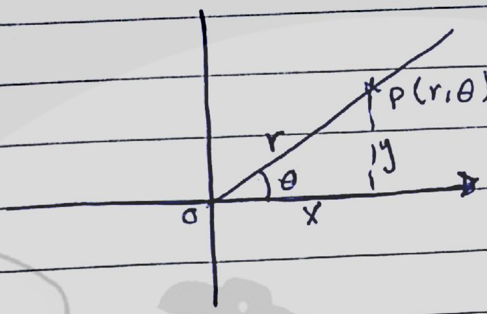
$(-3, \frac{\pi}{3}+\pi)$

$(-3, \frac{\pi}{3}-\pi)$

(r, θ) and $(-r, \theta+\pi)$

and $(-r, \theta-\pi)$ are equiveland

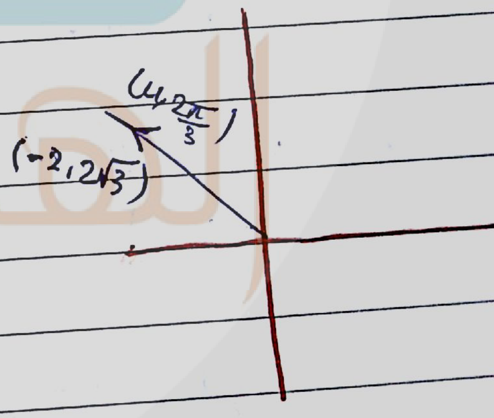
Change from polar coordinate to rectangular coordinates.



$x = r \cos\theta$

$y = r \sin\theta$

Example The coordinates of the point $(4, \frac{2\pi}{3})$ to rectangular coordinates.



$X = 4 * \cos\frac{2\pi}{3}$

$= 4 * -\frac{1}{2} = -2$

$y = 4 * \sin\frac{2\pi}{3}$

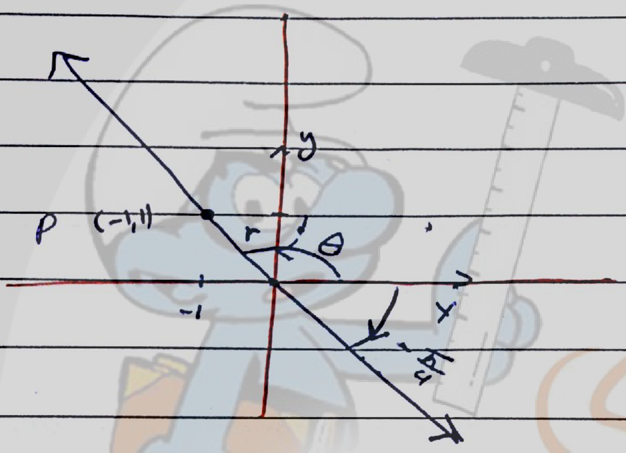
$= 4 * \frac{\sqrt{3}}{2}$

$= 2\sqrt{3}$

Change from rectangular to polar Coordinates.

Ex Change the Coordinates of the point $p(-1,1)$ from rectangular to polar Coordinates.

١ نرسم رسمة بسيطة نبين فيها النقطة



$$r^2 = x^2 + y^2 \quad \text{---} \ (1)$$

$$r^2 = (-1)^2 + 1^2 = 2$$

$$r = \mp\sqrt{2}$$

$$\tan\theta = \frac{y}{x} = -1 \quad \Rightarrow \quad \theta = -\frac{\pi}{4}$$

$$p\left(\sqrt{2},\ 3\frac{\pi}{4}\right)$$

or

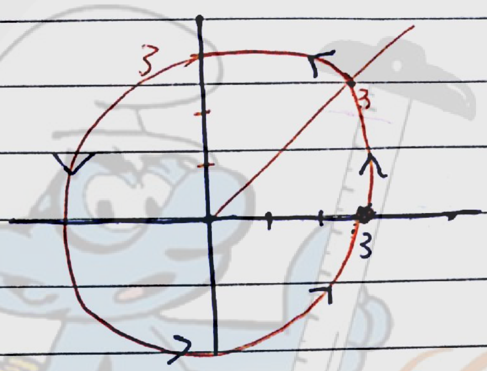
$$p\left(-\sqrt{3},\ -\frac{\pi}{4}\right)$$

Curves in polar Coordinates.

A Curve in polar Coordinates is a relation $f(r, \theta) = 0$

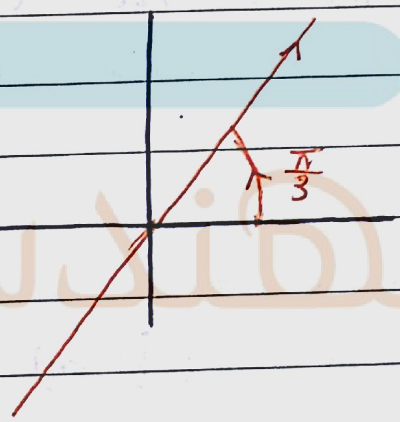
① $r = 3$

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	--
r	3	3	3	--



This is a circle with center $(0,0)$ radius $= 3$

② $\theta = \frac{\pi}{3}$

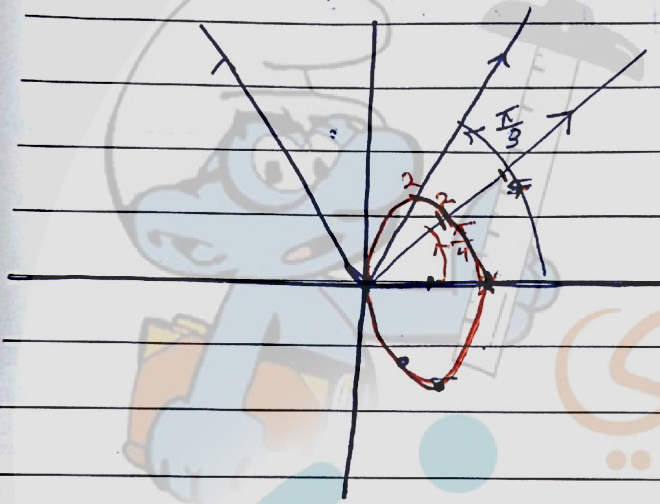


is the straight on lines that passes through ~~[illegible]~~ $(0,0)$ and slope $= \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$
whose equation in rectangular Coordinates in

$$\boxed{y = \sqrt{3}\, x}$$

$\boxed{3}$ $r = 2\cos\theta$

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	π
r	2	$\sqrt{2}$	1	0	-1	$-\sqrt{2}$	-2

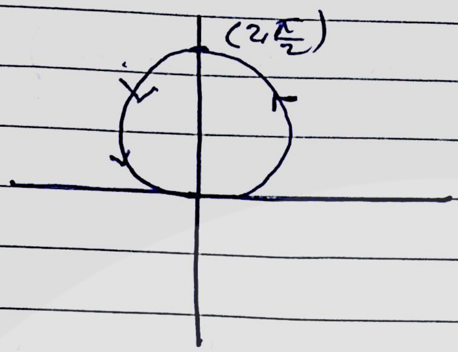


هو لازم يطلع دائرة

This is the circle while center as $(1,0)$ and radius $= 1$

$r = a\cos\theta$, $a > 0$ is a circle with center $(\frac{a}{2}, 0)$ and radius $= \frac{a}{2}$

* $r = 2\sin\theta$



This is the Circle with Center $(1, \frac{\pi}{2})$, radius $= 1$

*الفرق بين sin والـ cos ٩٠ درجة

* This graph can be obtained from $r = 2\cos\theta$ by rotating the graph of $r = 2\cos\theta$

$r = 2\cos\theta$ $\quad * r$

$r^2 = 2r\cos\theta \quad \rightsquigarrow \quad x = r\cos\theta$

$x^2 + y^2 = 2x \qquad y = r\sin\theta$

$x^2 - 2x + y^2 = 0$ نجعلها كامل مربع

$x^2 - 2x + 1 + y^2 = 1$

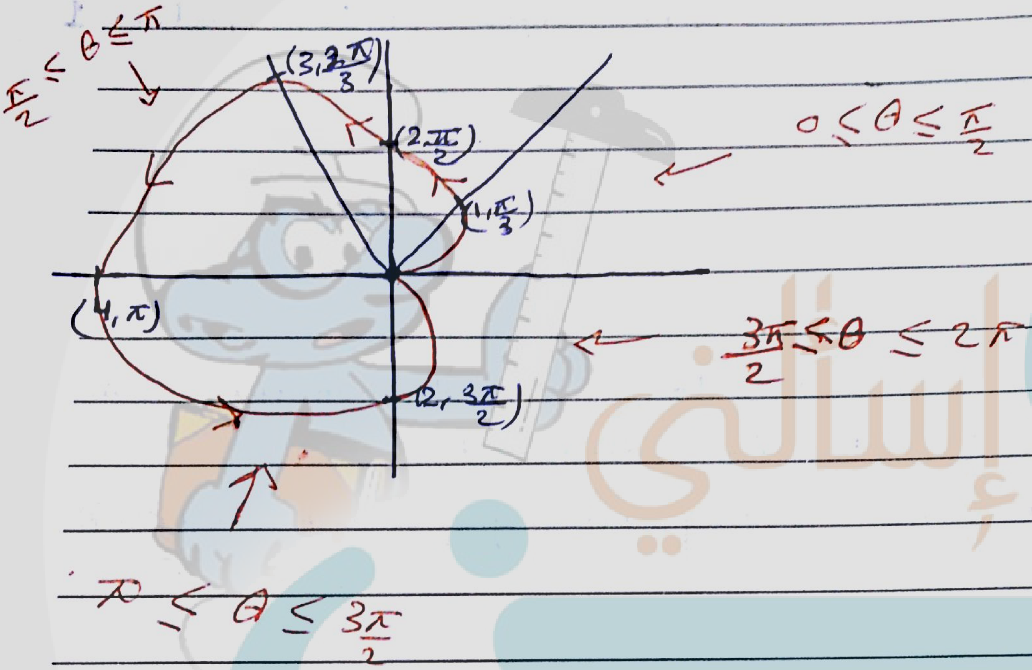
$(x-1)^2 + y^2 = 1$

$r = 2\sin\theta$

$\Leftarrow \Leftarrow x^2 + (y-1)^2 = 1$

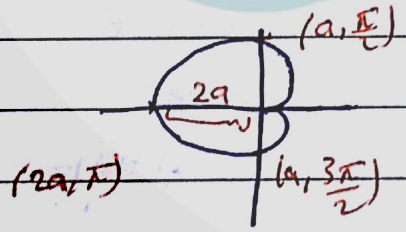
Cardioids
بشكل قلبي

$r = 2 - 2\cos\theta$ مثال $(r = a - a\cos\theta \ ; \ a > 0)$



شكل يعني يشبه القلب !!

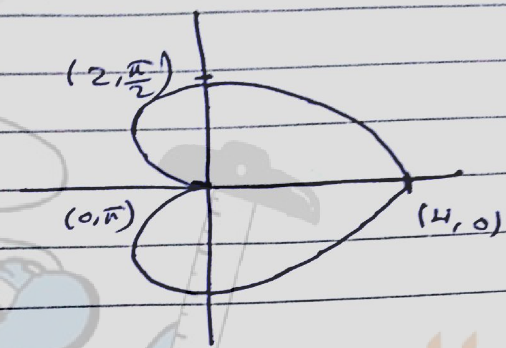
$r = a - a\cos\theta \ , \ a > 0$ بشكل عام



هيك رسمه <3

ملحوظة:
نقطة الأصل تمثل بعد لانهائي
مسافة

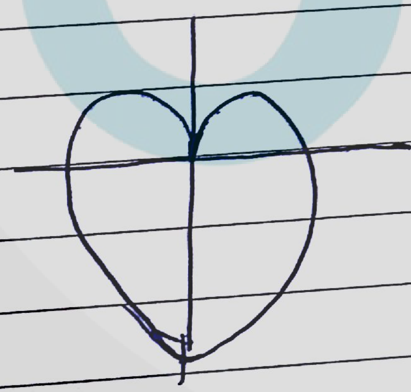
* $r = 2 + 2\cos\theta$



This can be obtain by reflecting the curve $y = 2 - 2\cos\theta$ in the line $\theta = \frac{\pi}{2}$

* $r = 2 - 2\sin\theta$

حلوة الشغلة !!



Ex

#سؤال سنوات#

Find the center and the radius of the circle.

$r = 2\cos\theta + 4\sin\theta$.

*هناك طريقتين للحل :-

① نأخذ قيم x,y ونعوضها حسب المطلوب

② نحوله لـ xy بإكمال المربع .

$r^2 = 2r\cos\theta + 4r\sin\theta$

$x^2 + y^2 = 2x + 4y$

$x^2 - 2x + 1 - 1 + y^2 - 4y + 4 - 4 = 0$

$(x-1)^2 + (y-2)^2 = 5$

This is a circle with center $(1,2)$ and radius $= \sqrt{5}$

Ex

Find the slope of the line.

$\cancel{r = 2\sec\theta + \frac{r}{\cos\theta}}$

$r = \dfrac{2}{\sin\theta}$

Sol

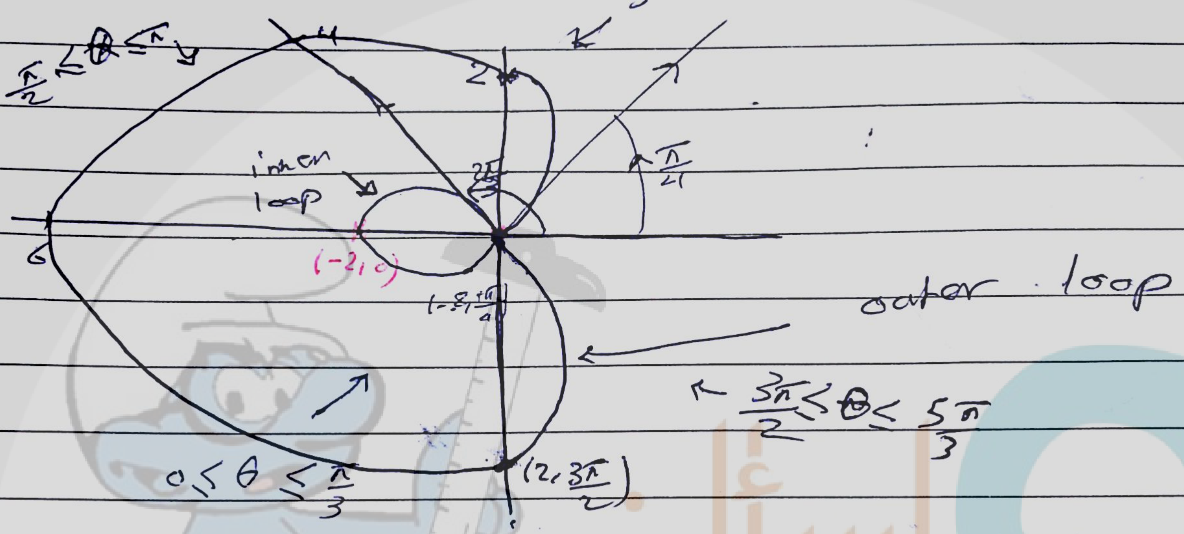
$r\sin\theta = 2$

$y = 2$ AND . خط موازي لمحور السينات

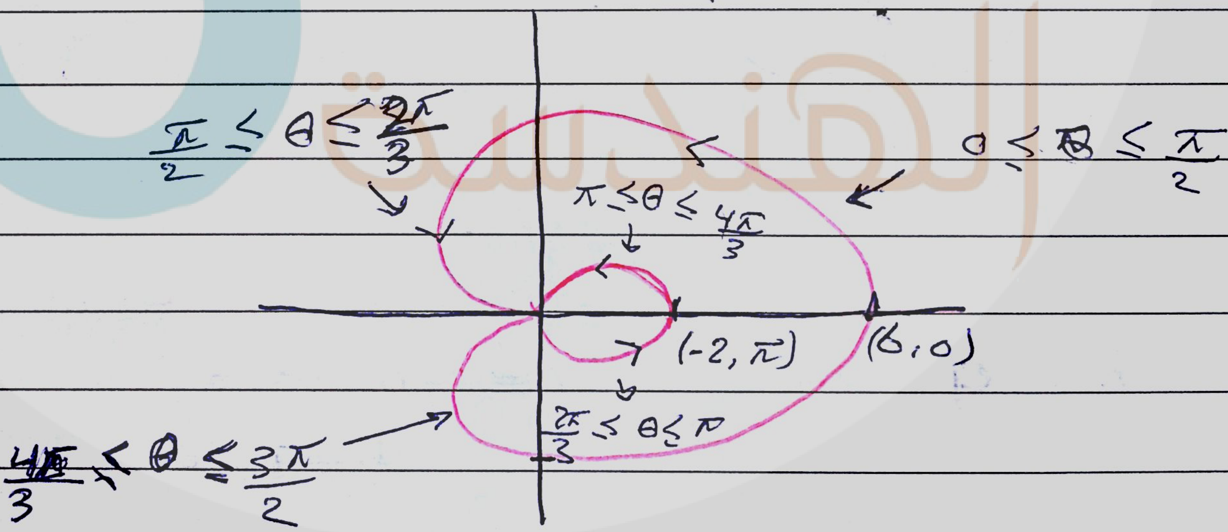
$\text{slop} = 0$

Limaçon

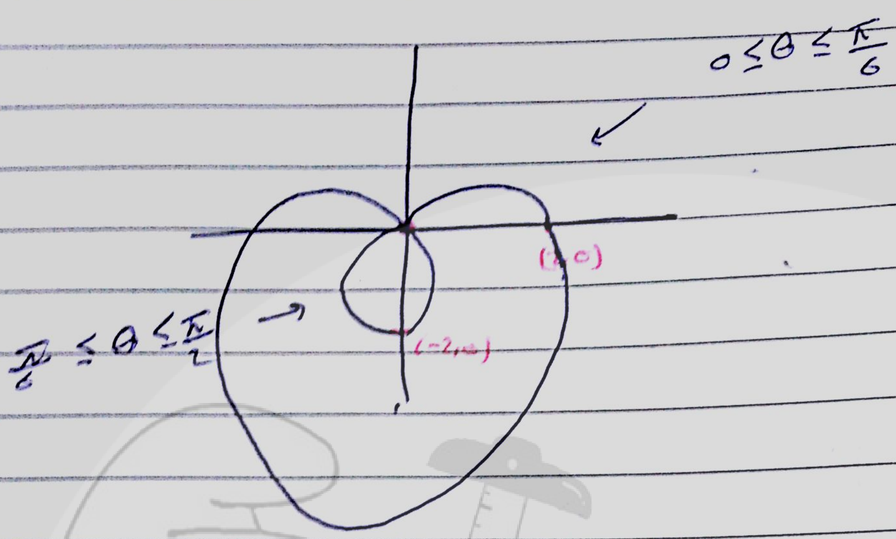
$r = 2 - 4\cos\theta$ $\quad \frac{2\pi}{3} \leq \theta \leq \frac{\pi}{2}$, $4 > 2$



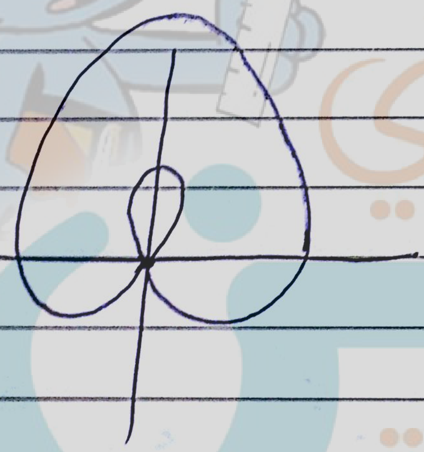
$r = 2 + 4\cos\theta$



* $r = 2 - 4\sin\theta$



* $r = 2 + 4\sin\theta$



* الصورة العامة

$(r = a - b\cos\theta)$

$b > a > 0$

$\cancel{b < a < 0}$

3. The integral in the form $\int \frac{p(x)}{Q(x)}\, dx$, $p(x)$. $Q(x)$ are polynomials , that equals the integral $\int \frac{x+5}{\sqrt{x}+\sqrt[3]{x}}\, dx$

$x = u^6$, $dx = 6u^5 du$

$$\int \frac{(u^6+5)}{u^3+u^2}(6u^5)\, du$$

4. The sum $\sum_{n=1}^{\infty} \frac{3^n}{n!}$ equals

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

$$\sum_{n=1}^{\infty} \frac{3^n}{n!} = e^3 - 1$$

بدك تكونها من الحال

5- Given $\frac{1}{1-x} = 1 + x + x^2 + \cdots$ $\quad -1 < x < 1$ 6

then the maclaurin series of $\ln|1+x|$. Is $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$

$$\ln|1+x| + C = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$ (بالتكامل)

$-1 < x < 1$

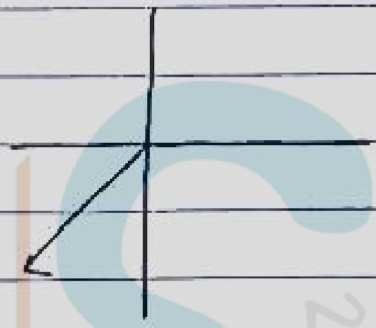
$C = 0$ $\Rightarrow$ from $x = 0$

Revision :-

1- Consider the point $(-\sqrt{3}, -1)$ in rectangular coordinates the corresponding polar coordinates are ?

$r^2 = x^2 + y^2$

$= 3 + 1 = 4 \rightarrow r = 2$

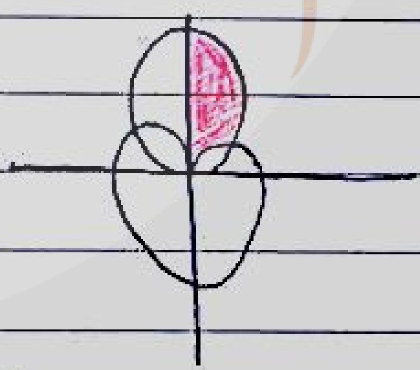


$y = r \sin\theta$

$-1 = 2\sin\theta \rightarrow \theta = \frac{\pi}{6}$

$(-2, \frac{\pi}{6})$ or $(2, 4\frac{\pi}{3})$ سهل !!

The integral that gives the area of the shaded region in polar coordinates is :-



$4\sin\theta = 4 - 4\sin\theta$

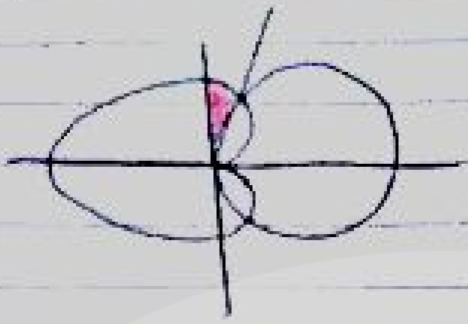
$\frac{8}{8}\sin\theta = \frac{4}{8} \rightarrow \theta = \frac{\pi}{6}$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{2}(4\sin\theta)^2 \, d\theta - \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{2}(4 - 4\sin\theta) \, d\theta$$

انتبه بعضهن بيطلب الا الناتج

المساحة

Ex



$r = 1 - \cos\theta$

$r = \cos\theta$

$1 - \cos\theta = \cos\theta$

$2\cos\theta = 1$

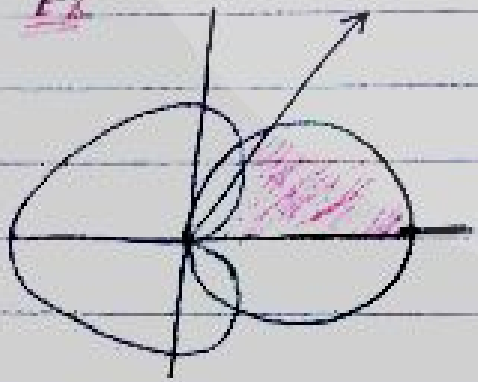
$\theta = \frac{\pi}{3}$

$$A = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{2}(1 - \cos\theta)^2 \, d\theta - \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{2}(\cos\theta)^2 \, d\theta$$

Ex

$$A = \int_{0}^{\frac{\pi}{3}} \frac{1}{2}(1 - \cos\theta)^2 \, d\theta + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{2}(\cos\theta)^2 \, d\theta$$

Ex



$$A = \int_{0}^{\frac{\pi}{3}} \frac{1}{2}(\cos\theta)^2 \, d\theta - \int_{0}^{\frac{\pi}{6}} \frac{1}{2}(1 - \cos\theta) \, d\theta$$

The limit of the sequence is $\left\{ n \sin\left(\frac{2}{n}\right) \right\}_{n=1}^{\infty}$ is

$= 2$ by L.H

Using the series $\sin(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^{(n-1)}}{(2n-1)!}$ $-\infty < x < \infty$

to find the sum of $\sum_{n=1}^{\infty} \frac{(-1)^n 5\pi^{n-1}}{6^{n-1}(2n-1)!}$

$= \cancel{5\pi} \quad = 5 \sin \frac{\pi}{6}$

$= \frac{5}{2}$

ملاحظات :-

- بالامتحان اذا كتبت maclaurin غلط بخصم علامات
- " لما تقول مقارنات بين series ب C.T أو L.C.T اذا ما حطيت Σ بخصم علامة .

* نصيحة مني يوم الامتحان ادعاء بكم يوم لا تدخل على الجروبات .. الا اذا بدك تسأل داخلي بسرعة لأنك رح تكتئب :)

\# اسألني-عن-الهندسة

\# دعواتكم :)

\# we-try-we-change. <3